gymnasium liestal

Matura Examination 2015 – Mathematics

Classes: 4GL, 4LW

Note:	You have four hours to complete the examination.
	Begin each question on a new sheet of paper.
Permitted materials:	TI-*n*spire CAS calculator (in 'press-to-test' mode)
	The *Fundamentum Mathematik und Physik*, without notes
	English-German dictionary

Question 1: Analysis

Consider the function

$$f(x) = \tfrac{1}{2}x^3 - \tfrac{3}{2}x^2$$

Parts a) und b) of this question should be answered "by hand".

a) Find the zeroes of the function f, as well as the coordinates of any local maxima, local minima and points of inflection. (4 P.)

b) Enclosed between the graph of f and the x-axis is a region A of finite area. Suppose we draw a vertical line through the local minimum of f, dividing A into two regions. Calculate the ratio[1] of the areas of these two regions. (3 P.)

Now consider the family of functions $\{f_p\}$, where for each non-zero, real number p,

$$f_p(x) = \tfrac{1}{2}x^3 - \tfrac{p}{2}x^2$$

In parts c) to e) you may use the full capacity of your calculators. However, make sure you still explain your answers clearly.

c) Show that each function in the family has a stationary point which does not lie at the origin. For which values of p is this point a local maximum? For which values of p is it a local minimum? (2 P.)

d) Find an equation for the curve containing the stationary points of all the functions in the family $\{f_p\}$. (1.5 P.)

e) For which values of p is the gradient of f_p equal to -2 at its point of inflection? (1.5 P.)

[1] ratio = *Verhältnis*

Question 2: Analysis

Consider the function

$$g(x) = \frac{4}{x^2}$$

a) Suppose we choose the corners of a rectangle as follows:

- One corner O lies at the origin $(0, 0)$.
- The corner diagonally opposite O lies on the right-hand branch of the graph of g.
- The sides of the rectangle are parallel to the coordinate axes.

Make a sketch of such a rectangle.
Which such rectangle has minimal perimeter[2]? Calculate both its width and its height. (4 P.)

b) Suppose we choose the corners of a triangle as follows:

- One corner O lies at the origin $(0, 0)$.
- One side of the triangle is tangent to the graph of g. This side intersects the x- and y-axes. These intersection points are the other two corners of the triangle.

Make a sketch of such a triangle.
Show that there is no such triangle with minimal area. (4 P.)

Now consider the part of the graph of g for which $x \geq 1$. Suppose we rotate this part of the graph around the x-axis to obtain a funnel-shaped object[3]. This funnel is bounded on the left at $x = 1$ but can be as long as we like on the right.

c) Where should we bound the funnel on the right so that its volume is equal to 15? (2 P.)

d) What is the largest volume that such a funnel could have? (2 P.)

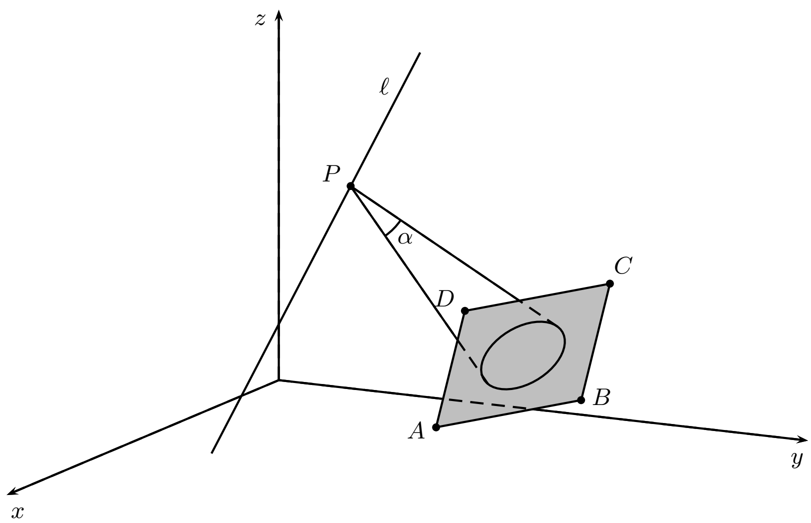
[2] perimeter = *Umfang*
[3] funnel = *Trichter*

Question 3: Vector Geometry

A light is shining onto a rectangular screen, producing a cone of light (see picture below). The corners of the screen are at the points $A(8, 20, 0)$, $B(-4, 26, 0)$, $C(-2, 30, 12)$ and $D(10, 24, 12)$. The light can be positioned at any point along a metal pole; the pole is represented by the straight line ℓ with vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -15 \\ 7 \\ 34 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

The angle of aperture[4] of the light, denoted below by α, can be varied as desired.



Suppose that the light is positioned at the point $P(-9, 1, 16)$.

a) Show that the point P lies on ℓ. (1 P.)

b) Calculate the area of the rectangular screen. (1 P.)

c) Find a Cartesian equation for the plane Γ which contains the screen. (2 P.)

If you cannot find a solution for part c), use the Cartesian equation $6x + 12y - 5z - 288 = 0$.

d) Calculate the distance between the point P and the plane Γ. (1 P.)

e) Show that the light shining on the centre of the screen hits the screen at right angles. (2 P.)

f) Suppose we open the angle of aperture α widely enough that the entire screen is illuminated. The screen will produce a shadow on the x-y-coordinate plane. Find the coordinates of C', the shadow of the point C. (2 P.)

g) Now suppose that we want to reduce the angle of aperture so that the illuminated region of the screen is as large as possible, but no light shines over the edges of the screen. Calculate the size of the appropriate angle α. (3 P.)

[4] angle of aperture = *Öffnungswinkel*

Question 4: Probability Theory

Your mathematics teacher has a collection of 24 standard questions about probability. Each time she wants to create a test, she chooses 5 of these questions.

a) How many different tests can be created this way? Assume that the order of the questions is irrelevant. (1 P.)

b) You estimate that for each question in the test, you have a chance of $\frac{2}{3}$ of finding the correct answer. What is the probability that in a test

   i. you answer exactly 3 of the questions correctly? (1.5 P.)

   ii. you answer at least 3 of the questions correctly? (1.5 P.)

In fact, among the 24 different questions, there are 8 easy, 8 medium and 8 difficult questions.

c) What is the probability that a test contains exactly one easy and four medium questions? Assume that your teacher chooses from the 24 questions at random. (2 P.)

Suppose from now on that the test always contains one easy, two medium and two difficult questions, each of which is chosen at random. To calculate your final mark, your teacher counts the number of correctly answered questions, and adds 1. In other words, to obtain the mark 4, 5 or 6 and therefore to pass the test, you need to solve at least 3 of the questions correctly.

d) How many different tests can be created now? (1.5 P.)

e) Suppose that you always answer easy questions correctly, but on average only three quarters of the medium questions correctly. With the difficult questions, you manage a correct answer only one out of four times. What is the probability that you obtain the mark 5 or 6? (3 P.)

f) For this part of the question, assume that the probability of obtaining a 6 is equal to $\frac{9}{256}$. How many times would you have to sit such a test for the probability of obtaining at least one 6 to be at least 90%? (1.5 P.)

Question 5: Trigonometry and the Exponential Function

a) John[5] ist an enthusiastic and innovative hang-glider pilot. He is planning to replace the control bar of his hang-glider by a new, slightly longer one. Before he goes ahead with the modifications however, he needs to make sure that the new design is safe to fly.

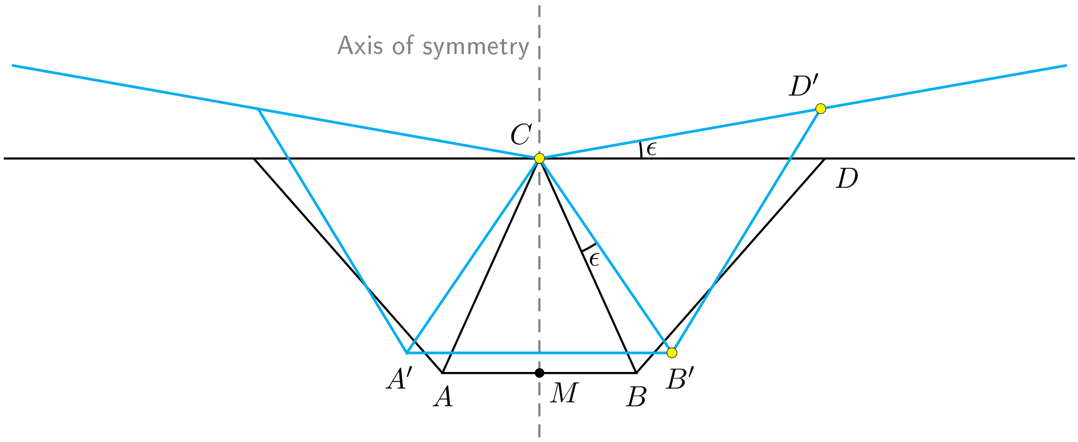
Shown in black below is the shape of the original hang-glider. The height of the triangular control frame ABC is $\overline{CM} = 1.60\,\text{m}$. The length of the original control bar is $\overline{AB} = 1.20\,\text{m}$. The length of the section $\overline{CD}$ is $3.15\,\text{m}$. (Note: the picture is not drawn to scale.)

Shown in blue below is the shape of the new hang-glider with a V-shaped wing. The new control bar $\overline{A'B'}$ is 6 cm longer than the old one. The lengths of all the other bars have not been changed.

To check the safety of the new wing design, John needs to answer the following questions:

i. What is the angle ϵ made by each of the wing halves with the horizontal?
ii. How big are the angles $\alpha = \angle D'B'C$ and $\beta = \angle B'CD'$?

Give all your answers correct to two decimal places. (6 P.)



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[5] The pilot in this question is named after the Australian inventor John Dickenson (*1934), who introduced a revolutionary new configuration for hang-gliders in 1963 that is still widely used today.

b) On his flights with the hang-glider, John uses a measuring device which displays the air pressure p in millibars as well as the altitude x in metres above the ground[6]. The quantities p and x are related by the formula

$$p = 978 \cdot \mathrm{e}^{-\frac{1}{8000}x}$$

where the number 978 represents the air pressure in millibars at the landing site and e is Euler's number.

i. Solve the equation above for x *"by hand"*. Show all your steps clearly. (2 P.)

ii. At which altitude is the air pressure equal to half of that at the landing site? (1 P.)

iii. At which altitude is the air pressure decreasing at a rate of 0.1 mbar/m? (1.5 P.)

iv. John is making his descent towards the landing site, losing 5 m in altitude each second. At an altitude of 1000 m he starts his stopwatch. What pressure is displayed on his measuring device 30 s later? Or asked more generally: what pressure is displayed on his measuring device t seconds later? (1.5 P.)

[6] altitude = *Höhe*