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Matura Examination 2013 – Mathematics (written)

Classes: 4(A)Z, 4GL, 4IS, 4ISW, 4LW, 4MW, 4S, 4W, 5KSW (BlT, HrP, KrD, LaG, PeM, PrG, RaM, ZuA)

Duration of Exam:	4 hours
Permitted Materials:	CAS – Calculator with its manual,
	Non-graphics capable/non-CAS- calculator as desired
	Formula Sheet (in English)
	Dictionary (English-German) kept on Teacher's Desk
	Fundamentum (in German)
Important Advice:	Start each question on a fresh sheet of paper.
	When using the calculator, stages of working should be written down.
	The maximum number of points to be obtained is given for each problem.

Vector Geometry (12 points)

1. A journey on a ray of light.

The idea of travelling on a ray of light comes from the famous scientist, Albert Einstein. "What would the world look like from this point of view?" is a question that, as a 16 year-old Gymnasium student, he was already asking himself . This seemingly harmless mental puzzle turned out later to be the seed of a truly revolutionary idea: the Theory of Relativity.

Our journey begins with a source of light at the point called $A(5|3|7)$. We then travel in the direction of the half line given by: $g : \vec{r} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, where $t \geq 0$.

(a) *This part-question must be fully solved by hand.*
At which point, B, would our flight path pass through the yz-plane?

(b) *This part-question must be fully solved by hand.* (1 P.)
Will our travel route pass through the point $C(-1|15|1)$? (1 P.)

(c) Investigate whether our path will cross that of a second light ray, h, which begins at the point $E(19|10|28)$ and sets out in the direction $\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$. If such a point, called D, exists, you should find its coordinates. (2 P.)

(d) We shall now consider the point $G(6|7|5) \notin g$, which does not lie on our travel path.

i. Which point, H, on the <u>half</u> line g is at a distance of 5 units from the point G? (2.5 P.)

ii. From which point, $I \in g$ would the distance to point G be the shortest? (2 P.)

(e) Journey's End: The destination of our journey is the plane $\epsilon : 4x + y + 3z - 52 = 0$.

i. At which point, J, will we reach the plane ϵ? (1 P.)

ii. What angle α will our flight path make with the plane ϵ? (1 P.)

(f) Explain in detail why the route which we have followed along the half-line g was not the shortest route from point A to reach the plane ϵ. Now calculate the length of the shortest route. (1.5 P.)

Probability (12 points)

2. A box contains 5 identical red balls, 3 identical white balls and 2 identical blue balls.

   (a) In how many different ways can these ten balls be arranged in a line? (1 P.)

   (b) Two of the balls are drawn from the box, one after the other, without replacement. What is the probability that these two balls have different colours? (1.5 P.)

   (c) Each ball drawn in this part-question has its colour recorded and is then immediately returned to the box.

      i. What is the probability that, after drawing ten balls, no blue balls have been recorded? (1 P.)

      ii. If four balls are drawn, what is the probability that exactly two of them are blue? (1.5 P.)

      iii. What is the smallest number of balls that we should draw in order to have a probability greater than 99% of recording at least one blue ball? (1.5 P.)

   (d) Three balls are drawn together from the box.
   What is the probability that exactly two balls have the same colour? (1.5 P.)

   (e) In the following game, players A and B take turns to draw a single ball, with replacement, from the box. Player A starts. The first player to draw a white ball wins the game. The game lasts for a maximum of four draws. If this fourth ball is not white then the game ends without a winner. What is the probability of winning for each of these players? (2 P.)

   (f) In this final part-question, the two blue balls are replaced with red balls. An unspecified number of white balls are now added to those already in the box. It is now given that when two balls are drawn without replacement from this box, the probability that they have different colours is exactly $8/15$. How many white balls were added? (2 P.)

Analysis 1 (12 points)

3. The family of functions f_k is defined as:
   $f_k(x) = x^3 - 2 \cdot k \cdot x^2 + k^2 x$ with the real parameter $k \geq 0$.

   (a) Calculate the zero points of f_k. (0.5 P.)

   (b) *This part-question must be fully solved by hand.*
   Calculate, in terms of k, the x-coordinate of the inflexion point for each of the graphs of f_k. (1.5 P.)

   (c) Find the full coordinates of all local maximum points and minimum points on the graph of f_k and identify their type. (2.5 P.)

   We shall work with the parameter value $k = 3$.

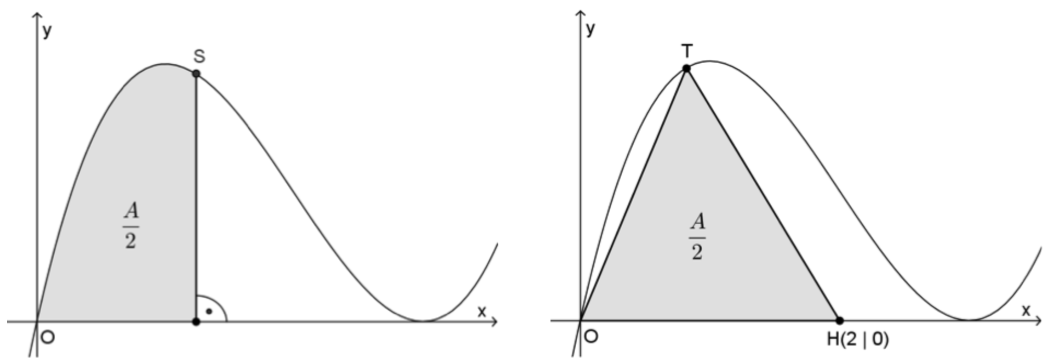
   The graph of the function $f_3(x)$ will be called G_3.

   (d) Show that G_3 cuts the line with equation $y = x$ at $x = 2$. Also, calculate the angle α (in degrees) between the given line and G_3 at this point. (2 P.)

   In the first quadrant, G_3 and the x-axis form a bounded region with finite area, A.

   (e) Show that $A = \frac{27}{4}$ area units. (1 P.)

   (f) Calculate the possible x-coordinates for the points S und T (see diagrams below). (2 P.)



   (g) An ant standing at the point $C(1|0)$ asks itself what the shortest distance would be to reach the curve G_3.
   Calculate the coordinates of the point $P \in G_3$, for which the distance between C and P is shortest. Hence, calculate this shortest distance. (2.5 P.)

Analysis 2 (12 points)

4. It is given that $f_t(x) = 2x \cdot e^{\left(\frac{t \cdot x}{2} - t\right)}$ is the equation of a family of curves with the real, finite parameter t.

   (a) *This part-question must be fully answered by hand.*
   Show that the point $(2\,|\,4)$ lies on all graphs of the family $f_t(x)$. (1 P.)

   (b) Calculate the coordinates of any maximum points. For which values of the parameter, t, do maximum points exist?
   (2.5 P.)

   (c) The graphs of the functions $f_{-2}(x)$ and $f_{-1}(x)$ create a bounded region, S, in the first quadrant. Calculate the area of this region S. (1.5 P.)

   (d) *This part-question must be fully answered by hand.*
   Show that $F_2(x) = (2x - 2) \cdot e^{x-2}$ is a possible integral function for $f_2(x)$.
   Hence, calculate the area between the curve of function $f_2(x)$ f2 and the x-axis for the interval $I_1 = [0; 2]$. (2 P.)

   (e) The graph of one of this family of functions $f_t(x)$ has a tangent with a gradient value of 3 at the point where $x = 2$. Calculate the value of the parameter, t, for this graph. (1.5 P.)

   (f) When $t < 0$ (*Do not forget this condition when working with the calculator*) the graph of $f_t(x)$ together with the x-axis form an unbounded region in the first quadrant.
   Firstly, show that this region has a finite area.
   Which value of parameter, t, gives us the smallest area for this region?
   Finally, calculate the 'smallest' area for this region. (2 P.)

   (g) Over the interval $I_2 = [\,0\,;\,p\,]$, the area under the graph of function $f_{-2}(x)$ is rotated 360° around the x-axis, creating a 3-D solid. For which value of p does this solid have a volume of 170 cubic units? (1.5 P.)

Short Problems (12 Points)

5. This problem is comprised of three independent parts. Use a *By Hand* method to answer all parts of this question.

   (a) The Cartesian equations of three different planes are:

   $$\Pi_A : 2x - 2y - z - 3 = 0$$
   $$\Pi_B : x + y - z + 7 = 0$$
   $$\Pi_C : k \cdot x - 2y + 2z - 13 = 0$$

   i. Find the meeting point of these three planes when k equals -3. (2.5 P.)

   ii. A. Show that the following system of equations has no solutions: (1 P.)

   $$\begin{aligned} 2x - 2y - z - 3 &= 0 \\ x + y - z + 7 &= 0 \\ -2x - 2y + 2z - 13 &= 0 \end{aligned}$$

   B. Give a geometric interpretation for this result. (0.5 P.)

   (b) The quadratic equation: $y = \frac{9}{2}x - x^2 - 3$ produces a curve which contains the following three named points:

   Point P : Where the x coordinate has the value of 2

   Point Q : Where the curve intersects with the y-axis

   Point R : Where, t_P, the tangent line to the curve at point P intersects with the y-axis.

   i. Calculate the full coordinates of these three points. (2 P.)

   ii. Calculate the area of the bounded region between: (2 P.)

   1: the tangent line, t_P,

   2: the curve of the quadratic equation $y = \frac{9}{2}x - x^2 - 3$ and

   3: the line, l_{QR}, which joins the points Q and R.

(c) A bag contains 12 coloured balls of which 3 are Blue, 4 are Red and 5 are Green.

David, Matthew and Sarah are offered the chance to play a guessing game.

They are told that a box contains Red, Green and Blue balls, but not how many of each!

They may guess either (but not both) of: the colour of the first ball picked out or the colour of the second ball picked out.

If they decide to guess the colour of the second ball picked out, then they will be shown the colour of the first ball picked, but it will not be replaced in the box.

i. Do you agree or disagree with the following calculation?
P(David correctly guesses the colour of the 1st ball)
$$= 1/3 \cdot 3/12 + 1/3 \cdot 4/12 + 1/3 \cdot 5/12 = 1/3$$
Give a clear explanation for your decision. (1 P.)

ii. Matthew's strategy is to watch the colour of the first ball picked and then choose the same colour for the second pick.
Sarah's strategy is to watch the colour of the first ball picked and then to toss a coin to decide between the two colours which did not appear.
Which of the three players has the best strategy? (1.5 P.)

iii. In a gambling game, the same 12 balls are used.
Rules:

1. To play this game, Matthew must pay amount P.

2. If Matthew selects a ball from the bag without looking, he can win the following amounts:
10 Francs if he picks a Green ball, 25 Francs if he picks a Red ball and 50 Francs if he picks a Blue ball.

3. The selected ball is always replaced immediately in the bag.

After playing this game several hundred times, he observes that he is making - on average - a loss of 2 Swiss Francs per game. Estimate the value of P, that Matthew must pay in order to play each game. (1.5 P.)