Question 1:

12 points

Given the points A(12 / -6 / 2), B(10 / 2 / 0), and C(4 / 2 / 6).

a) Show that the points A, B and C are the corners of a right-angled isosceles triangle.

b) Calculate the coordinates of the point D such that the points A, B, C and D are the corners of a square.

c) A plane E contains the square ABCD. Find a Cartesian equation for that plane E.

The points A, B, C and D mentioned above are vertices of an octahedron $ABCDS_1S_2$ (cf. figure 1). An octahedron is a regular solid whose surface consists of eight congruent equilateral triangles.
As shown in figure 1, the octahedron $ABCDS_1S_2$ is inscribed in a cube; each vertex of the octahedron lies at the centre of a face of the cube.

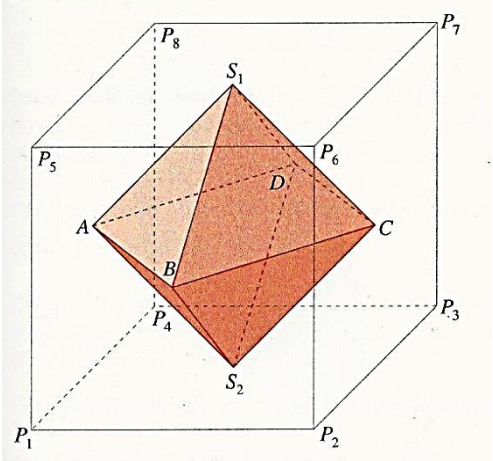


figure 1: Octahedron $ABCDS_1S_2$.

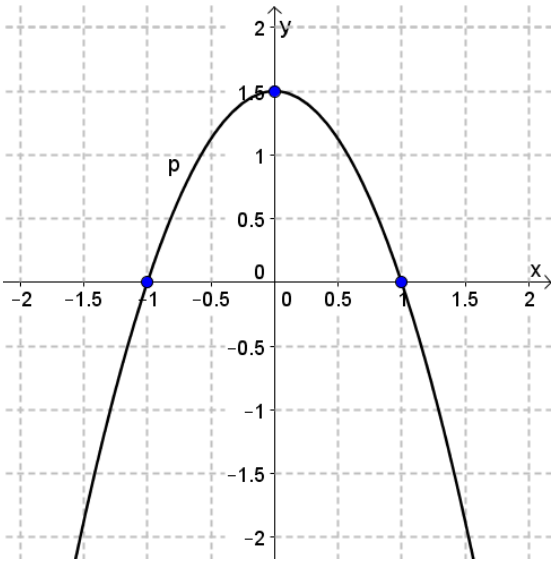
d) Calculate the coordinates of vertices S_1 and S_2 of the octahedron.

e) Calculate the coordinates of corner P_6 of the cube.

Question 2:

10 points

Given the graph of a quadratic function p.



a) Find a function equation for p. The points with a round mark may be read from the graph.

We now turn to a new function f with equation $f(x) = ax^3 + bx^2 + cx + d$, about which we know the following:

- The derivative of f is p.
- The graph of f has a point of inflexion at W(? / 1).
- The graph of f has a low point at T(? / 0).

b) Calculate the x-coordinates of points W and T. Give reasons for your answer!

c) Calculate a function equation for f.

In case you were not able to solve c), continue with the following (incorrect) function equation:

$$f(x) = -\frac{1}{3}x^3 + x + \frac{4}{3}$$

(Please note that the coordinates of the point of inflexion, W, are no longer the same as calculated in b)! You will need to calculate them anew.)

d) A straight line passing through the point of inflexion W intersects the graph of f at the position $x = u$ $(u > 0)$. Show that the line intersects the graph of f at the position $x = -u$ as well.

Question 3:

13 points

Five friends, Anna, Ben, Carl, Daniel and Emily, make it a habit to meet on Tuesday evenings at their favourite pub. Each person may or may not be able to join on a given Tuesday. In particular, it may also happen that their regulars' table ("Stammtisch") remains empty.

a) How many possible compositions of people at the regulars' table are there?

b) How many possible compositions are there, in which the number of men is the same as the number of women?

c) Explain in words: There are at least two Tuesdays in every year on which the composition of people at the regulars' table is exactly the same.

The following list shows the probabilities for each of the five friends to be at the regulars' table. The presence or absence of any person is independent from the presence or absence of everybody else.

Person	Anna	Ben	Carl	Daniel	Emily
is present with probability	0.8	0.5	0.4	0.8	0.5

d) What is the probability that all five of them are present?

e) There are four Tuesdays in January 2016. Calculate the probability that, during the whole of January 2016, there is no encounter between Daniel and Emily at the regulars' table.

When it comes to picking up the bill, the five have made the following agreement: Whoever comes first in alphabetic order among all those present pays the bill.

f) Carl goes to the regulars' table tonight. What is the probability that he will have to pay?

g) Carl goes to the regulars' table and learns that neither Anna nor Ben will be there. What is the expected number of people for whom he will have to pay the bill?

Question 4:

13 points

Given the function f with equation

$$f(x) = e^{x-1}$$

a) Calculate the coordinates of the point of intersection Q of the graph of f and the y-axis.

b) Find an equation for the tangent t touching the graph of f at point B(2 / ?).

c) The graph of f and the tangent from b) and the two coordinate axes enclose a finite region in the first quadrant. Calculate the area of that region. Give the exact value!

d) The parabola with equation $y = ax^2 + c$ intersects the graph of f perpendicularly at the point P(1 / ?). Calculate the values of a and c.

In case you were not able to solve d), continue with the (incorrect) function equation

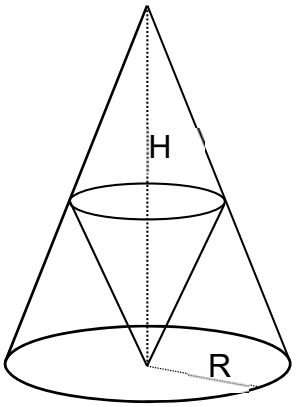
$$y = -\frac{1}{3}x^2 + \frac{4}{3}$$

e) What point on the parabola in the first quadrant lies closest to the origin of the coordinate system?

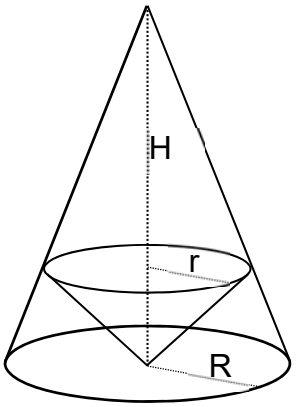
Question 5:

12 points

A right circular cone with radius R = 2 and height H = 6 has another circular cone inscribed, so that the tip of the inner cone lies in the centre of the base circle of the larger cone (see figure).



a) Calculate the lateral surface area ("Mantelfläche") of the inner cone, for the case in which the inner cone has half the height of the larger cone. Give the exact value!



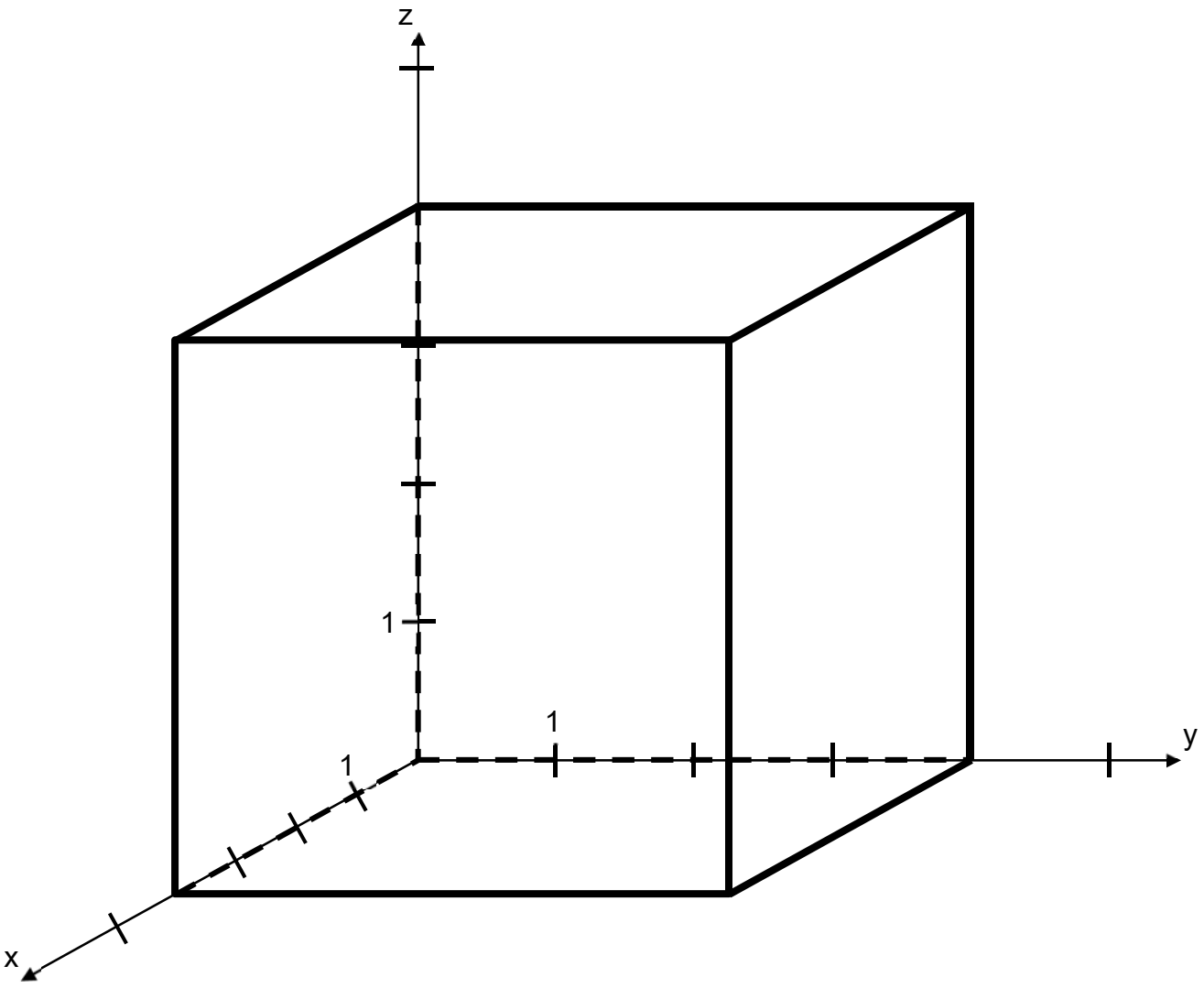
b) Show how you deduce the function expressing the lateral surface area M of the inner cone in terms of the radius r of the inner cone.
(For reference: the desired function is $M(r) = \pi \cdot r\sqrt{10r^2 - 36r + 36}$)

c) What is the maximum lateral surface area M the inner cone can have?

Short Question 6.1:

3 points

Given the plane E: $x + y + z = 5$ and a cube with edge 4. One corner of the cube lies in the origin. Three of its edges lie on the x-, y- and z-axes respectively. Colour the surface of intersection of the plane E with the cube.



Short Question 6.2:

3 points

A regular dice is rolled twice. The result of the first roll determines the value of the parameter m, and the result of the second roll determines the value of the parameter q, which make up the line equation $y = mx + q$. What is the probability that the "random line" thus obtained passes through the point P(2 / 7)?

Short Question 6.3:

2 points

The trajectories of two airplanes A and B, which are moving along straight lines, are described by the lines g_A and g_B, where $t \in \mathbb{R}$ ($t \geq 0$) denotes time, measured in minutes.

$$g_A\colon \vec{r} = \begin{pmatrix} -21 \\ -24 \\ 8 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$$

$$g_B\colon \vec{r} = \begin{pmatrix} -49 \\ -42 \\ 12 \end{pmatrix} + t \cdot \begin{pmatrix} 10 \\ 8 \\ -1 \end{pmatrix}$$

Neglect the dimensions of the airplanes. Check whether or not the two airplanes collide.