

Mathematics

Please start each problem at the top of a new page!

Time: 4 hours

Resources: Wetzell formulary
Calculator TI-83, TI83+, TI84+, TI84+ SilverEdition

Grading: For each problem the maximum attainable number of points is stated.
You do not need to reach the maximum number of points in order to get the grade 6.

1. Given the function f with equation

16 points

$$f(x) = \frac{1}{8}(x+2) \cdot (x-4)^2 = \frac{1}{8}x^3 - \frac{3}{4}x^2 + 4$$

- Find all extremal points and points of inflexion of the graph of f .
- Calculate the area of the region enclosed by the graph of f and the x -axis.
- The point P has x -coordinate 6 and belongs to the graph of f .
Find an equation for the normal line to the graph of f , passing through P .
- The graph of f , the normal line from part c) and the x -axis enclose a region.
Calculate the area of this region.

We shall now consider another function g with equation

$$g(x) = \frac{1}{8}x^3 - \frac{7}{8}x^2 - \frac{1}{8}x + \frac{19}{4}.$$

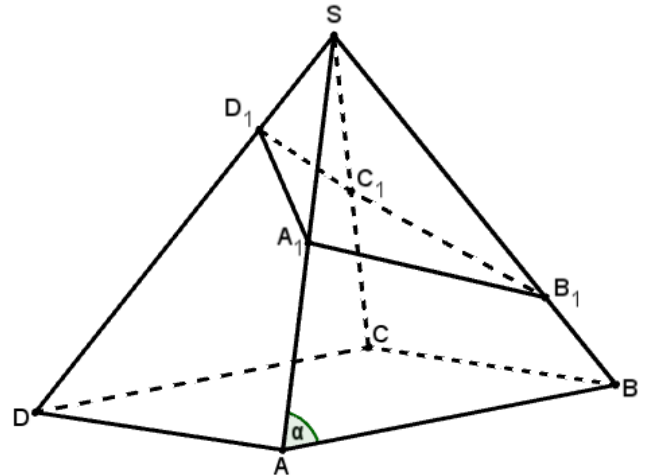
- Calculate the points of intersection of the graphs of f and g .
- The vertical line with equation $x = a$ intersects the two graphs at points R and S . What is the value of a , if the two tangents at R and S are parallel?

2. Given a right pyramid with square base $ABCD$ and apex S :
 $A(6 / 0 / 0)$, $B(0 / 6 / 0)$, $C(-6 / 0 / 0)$, $D(0 / -6 / 0)$ and $S(0 / 0 / 8)$.

9 points

Each of the points A_1 , B_1 , C_1 and D_1 lies on another edge of the pyramid. The points are at different heights above the ground surface: B_1 at height 2, both A_1 and C_1 at height 4, and D_1 at height 6.

- Calculate the coordinates of the points A_1 , B_1 , C_1 and D_1 .
- Calculate the angle α in the triangle ABS .
- Show that the points A_1 , B_1 , C_1 and D_1 do not all lie in the same plane.
- What are the coordinates of another point D_2 , still on the edge DS , such that A_1 , B_1 , C_1 and D_2 lie on a common plane?
- We now examine the quadrilateral $A_1B_1C_1D_2$. What kind of quadrilateral is it? Give reasons for your answer.



3. We consider two functions f and g with equations

$$f(x) = \ln(x) \quad \text{and} \quad g(x) = 2 \cdot \ln\left(\frac{x}{2}\right)$$

12 points

- Sketch both graphs, using the same coordinate system. Calculate the coordinates of the point of intersection of the two curves.
- Let P be a point on the graph of g . O is the origin of the coordinate system. Show that the midpoint M of the line segment OP lies on the graph of f .
- From O , a tangent to the graph of f is drawn. Show that this tangent also **touches** the graph of g at $x = 2e$.
- Let $Q(u / ?)$ be a point on the graph of f . Find the equation of the tangent t which touches the graph of f at Q .
- The tangent t from part **d)**, the horizontal line through Q , and the y -axis form a right-angled triangle. For what value of Q does this triangle have an area of 5?

4. Given the points $A(7 \mid -3 \mid 8)$ and $B(-1 \mid 5 \mid 4)$.

14 points

The line segment AB is the base of an **isosceles** triangle ABC .

- a) In this part of the problem, C lies on the line g defined by the points $P(2 \mid 0 \mid 3)$ and $Q(3 \mid 2 \mid 6)$.
- Determine the coordinates of point C .
 - Calculate the area of the triangle ABC .

(In case you were unable to find C : calculate the area of the triangle ABP instead.)

In parts b) and c), the point C belongs to the xy -plane.

- b) In the xy -plane, determine the curve of all possible places where C might be. Give an equation for that curve.

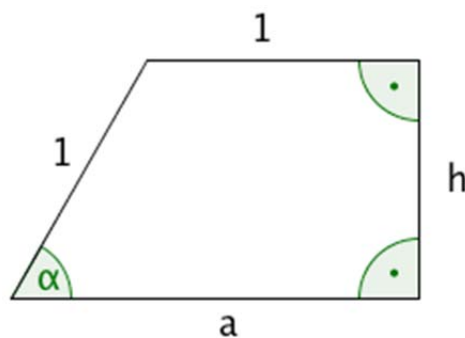
Which of these possible points C is closest to the origin O ?

- c) Which one of these points C in the xy -plane minimises the area of the triangle ABC ?

5. A trapezium is known to have two sides of length 1 and two angles of 90° as shown in the figure.

9 points

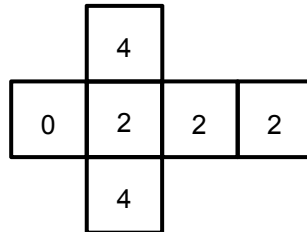
Determine a , h and the angle α such as to maximise the area of the trapezium.



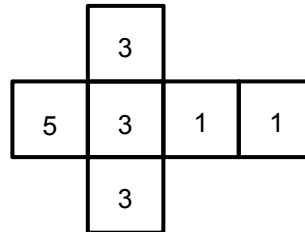
6. We consider two dice A and B whose nets are shown here:

13 points

Die A:



Die B:



- a) Die A is rolled once.
 - i) What is the probability for a '4'?
 - ii) What average number is to be expected?
- b) Die B is rolled four times.
 - i) What is the probability of the number '1' occurring at least twice?
 - ii) What is the probability that the four numbers add up to 8?
- c) Two people play the following game: The first player gets to choose one of the above dice, the other player has to take the other die. Both players roll their die once. The one with the higher score wins.
 Assume you are the first player to choose a die. Which die would you choose? Justify your answer by a calculation.
- d) An observer suspects that the number '5' on die B appears too often. You decide to put this assumption to the test by rolling the die 400 times. What is the least number of '5's that would lead you to believe that the die has been manipulated? Justify your answer (using a significance level of 2.25%).